

# 7 - Choice of Approximating Function

7.1) Consider the 1D-mesh  $\Rightarrow$

a) Determine the element shape functions.

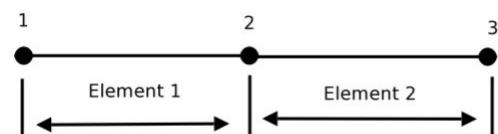
$$\phi = N_1^1 + N_2^2$$

$$N_1^1 = 1-x, \quad N_2^2 = x$$

Intuitive solutions

$$N_1^2 = 2-x, \quad N_2^1 = -1+x$$

X=0 X=1 X=2



X

b) Determine the global shape functions.

Definition, see page 103:

$$N_1 = \begin{cases} N_1^1, & \text{in element 1} \\ 0, & \text{in element 2} \end{cases}$$

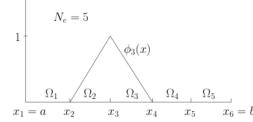
$$N_2 = \begin{cases} N_2^1, & \text{in element 1} \\ N_2^2, & \text{in element 2} \end{cases}$$

$$N_3 = \begin{cases} 0, & \text{in element 1} \\ N_2^2, & \text{in element 2} \end{cases}$$

2 of 93

Example: linear elements

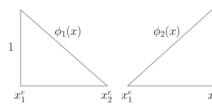
Global shape functions:



$$u_h(x) = \sum_{i=1}^n u_i \phi_i(x) = \phi^T(x) u$$

n: number of global nodal points.

Local element shape functions:



$$u_h^e(x) = u_1^e \phi_1(x) + u_2^e \phi_2(x) = \phi^T(x) u^e$$

with  $\phi^T = [\phi_1(x), \phi_2(x)]$  and

$$\phi_1(x) = \frac{x - x_2^e}{x_1^e - x_2^e}, \quad \phi_2(x) = \frac{x - x_1^e}{x_2^e - x_1^e}$$

Let us try the C-matrix method.

$$N = [1 \ x]$$
 (degree 1)

Element 1

$$C = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

coordinates for nodes

$$T = \bar{N} \bar{\alpha} \text{ and } \bar{\alpha}^e = \bar{C} \bar{\alpha} \Rightarrow T = \bar{N} \cdot \bar{C}^{-1} \bar{\alpha}^e = N^e \alpha^e$$

We are looking for T, so we need the inverse of C.

$$C^{-1} = \frac{1}{\det(C)} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \Rightarrow T = [1 \ x] \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \alpha^e = \boxed{[1-x, x]} \alpha^e$$

Here are our element functions for element 1.  
 $N_1^1$   $N_2^2$

Element 2

$$\bar{N} = [1 \ x], \quad C = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow C^{-1} = \frac{1}{1} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \Rightarrow T = [1 \ x] \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \alpha^e = \boxed{[2-x, -1+x]} \alpha^e$$

$N_1^2$   $N_2^1$

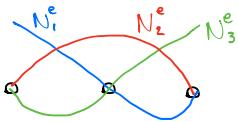
b) Global shape function?

We need one global shape function for each node. You only care about the shape functions that  $\neq 0$  in the node!

$$\text{Node 1: } N_1 = \begin{cases} N_1^1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}, \quad N_2 = \begin{cases} N_2^1 & 0 \leq x \leq 1 \\ N_2^2 & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}, \quad N_3 = \begin{cases} N_2^2 & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

7.2) Consider the 1D-mesh

a) Determine the element shape functions.

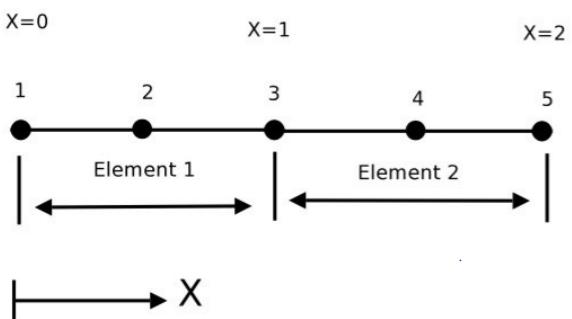


$$N_1^e = 2(x - 0.5)(x - 1)$$

$$N_2^e = 4x(x - 1)$$

$$N_3^e = -2x(0.5 - x)$$

Intuitive solution



b) Determine the global shape functions.

See definition on page 103.

$$N_1 = \begin{cases} N_1^e & \text{in } \Omega_1 \\ 0 & \text{in } \Omega_2 \end{cases}, N_2 = \begin{cases} N_2^e & \text{in } \Omega_1 \\ 0 & \text{in } \Omega_2 \end{cases}, N_3 = \begin{cases} N_3^e & \text{in } \Omega_1 \\ N_1^e & \text{in } \Omega_2 \end{cases}, N_4 = \begin{cases} 0 & \text{in } \Omega_1 \\ N_2^e & \text{in } \Omega_2 \end{cases}, N_5 = \begin{cases} 0 & \text{in } \Omega_1 \\ N_3^e & \text{in } \Omega_2 \end{cases}$$

7.3) 6- and 9-node elements.

a) Check if the convergence criterion is fulfilled.

6-node

$$T_6 = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy + \alpha_5 x^2 + \alpha_6 y^2$$

9-node

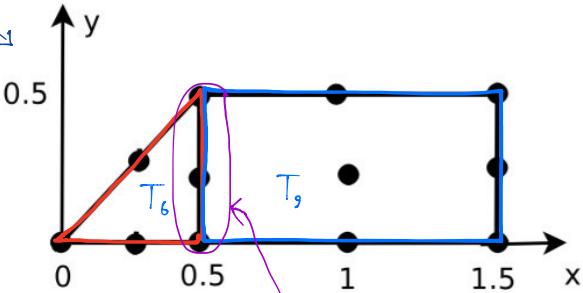
$$T_9 = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 x^2 + \beta_5 y^2 + \beta_6 xy + \beta_7 x^2y + \beta_8 xy^2 + \beta_9 x^2y^2$$

Ok, so how can we check if T converges?

Convergence = Completeness + Compatibility

Compatibility:  $T_6 = T_9$  on the boundary between the shapes.

Let us look at the line marked in purple ( $x = 0.5$ ,  $y: 0 \rightarrow 0.5$ ),



**Completeness:** The temperature functions  $T_6$  and  $T_9$  can represent the temperature for constant arbitrary  $T$ -values and  $\nabla T$ -values.

Constant  $T$  value is OK, just go through all nodes and "spend" one constant in each node.

The same holds for  $T_9$ !

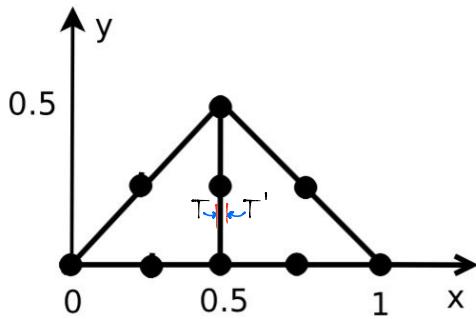
$\Rightarrow$  The convergence criterion is satisfied.

b) Check if the convergence criterion is satisfied.

$$T = \alpha_1 + \alpha_2 X + \alpha_3 Y + \alpha_4 XY + \alpha_5 X^2 + \alpha_6 Y^2$$

Interesting part: ( $x=0.5, y: 0 \rightarrow 0.5$ )

Same thing here,  $\alpha_i$  can be chosen such that the temperature is continuous, ( $T = T'$ ).



$\Rightarrow$  The convergence criterion is satisfied.

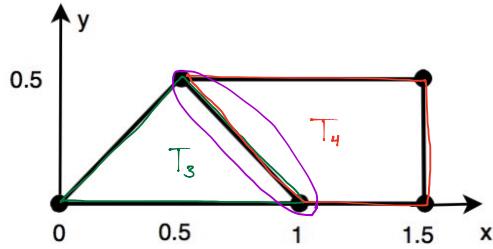
### 7.4) Element mesh of 3- and 4-node elements.

$$T_4 = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$

$$T_3 = \beta_1 + \beta_2 x + \beta_3 y$$

a) Is the convergence criterion fulfilled?

Completeness: OK, I think.



Compatibility: Let's look at the  $(x: 0.5 \rightarrow 1, y = 1-x)$  line.

$$\left\{ \begin{array}{l} T_4 = \alpha_1 + \alpha_2 x + \alpha_3 (1-x) + \alpha_4 x(1-x) \\ T_3 = \beta_1 + \beta_2 x + \beta_3 (1-x) \end{array} \right.$$

Since  $T_4$  is of degree 2 and  $T_3$  is of degree 1, it is impossible to satisfy  $T_4 = T_3$  over the entire line  $\Leftrightarrow T_4 \neq T_3$  for  $x \in [0.5, 1]$ .

b) Use the C-matrix method to obtain the element shape functions for the 3-node element.  
Remember that the nodes are numbered in positive direction!

$$N = [1 \ x \ y], C = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0.5 & 0.5 \end{bmatrix} \Rightarrow \det(C) = +0.5$$

$$\text{MATLAB: } C^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 2 \end{bmatrix} \quad T = \bar{N} \cdot C^{-1} \alpha^e = [1 \ x \ y] \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 2 \end{bmatrix} \alpha^e = \boxed{(1-x-y \ x-y \ 2y)} \alpha^e$$

Answer!

7.5) One 9-node element & one 6-node triangular element.

$$\mathbf{a}^T = [13 \ 15 \ 18 \ 14 \ 14 \ 16 \ 13 \ 15 \ 18 \ 14 \ 14 \ 19]$$

$$Coord = \begin{bmatrix} 0 & b & 2b & 3b & 4b & 0 & b & 2b & 3b \\ 0 & 0 & 0 & 0 & 0 & b & b & b & 2b \end{bmatrix}$$

Determine the total heat generated within the structure due to the chemical reaction for the situation

where the thickness is  $b$  and the ambient temperature,  $T_\infty = 0$ .

$\nabla \cdot \mathbf{q} - Q = 0$ , Determine  $Q$ .

$\bar{q}$  is the flux.  $\bar{q} = -\bar{D}\bar{\nabla}T = -K\bar{I}\bar{\nabla}T$

$T = \bar{N}^e \bar{\alpha}^e$ , where  $\bar{\alpha}^e$  is given.  $\bar{\alpha}^e = \bar{\alpha} = [13 \ 15 \dots 19]^T$

$$\int Q dA = \int q_n dL = \int_{L_{5-12}} \alpha(T - T_\infty) dL = \int_{x=5}^{x=3} \int_{y=4b-x}^{y=0} \alpha(T - T_\infty) dx dy = \int_{x=5}^{x=3} \alpha(T - T_\infty) \sqrt{1 + (\frac{dy}{dx})^2} dx = \int_{x=5}^{x=3} \alpha(T - T_\infty) \sqrt{2} dx = \boxed{\int_{x=5}^{x=3} \alpha(T - T_\infty) \sqrt{2} dx}$$

So what is  $T = T(x, y)$  along  $L_{5-12}$ ?

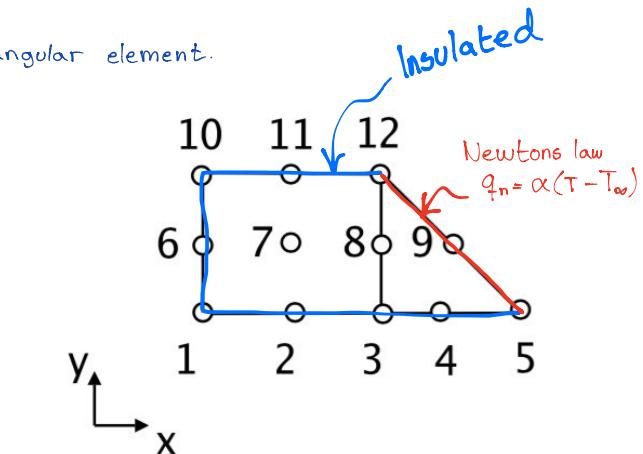
$$T(5) = 14, T(9) = 18, T(12) = 19$$

Ansatz:  $T = \alpha_1 + \alpha_2 x + \alpha_3 y$

$$T(5) = \alpha_1 + \alpha_2 \cdot 4b + \alpha_3 \cdot 0 = 14$$

$$T(9) = \alpha_1 + \alpha_2 \cdot 3b + \alpha_3 \cdot b = 18$$

$$T(12) = \alpha_1 + \alpha_2 \cdot 2b + \alpha_3 \cdot 2b$$



$$T(x) = \alpha x^2 + bx + c$$

$$T(5) = 8\xi^2 + 2b\xi + c = 14$$

$$T(9) = 18\xi^2 + 3b\xi + c = 18$$

$$T(12) = 32\xi^2 + 4b\xi + c = 19$$

$$8\xi^2 + 2b\xi + c = 14$$

$$0 - 1,5b\xi - 3,5c = -13,5$$

$$0 - 4b\xi - 3c = -37$$

7.6)

a) Suggest  $\phi$  that fulfills the convergence requirement

Let's pick these elements!

$$\phi = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 x^2y$$

$x^4 x^3 y x^2 y^2 x y^3 y^4$  Begin from the middle on row 4 since we are equally interested x- and y-dependence.

$x^5 x^4 y x^3 y^2 x^2 y^3 x y^4 y^5$

$x^6 x^5 y x^4 y^2 x^3 y^3 x^2 y^4 x y^5 x y^6$ .

Checking the convergence criterion.



b) Same, but for a 6-node element.

We only need the first three rows!

$$\phi = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2$$

$x^4 x^3 y x^2 y^2 x y^3 y^4$

$x^5 x^4 y x^3 y^2 x^2 y^3 x y^4 y^5$

$x^6 x^5 y x^4 y^2 x^3 y^3 x^2 y^4 x y^5 x y^6$

better

Checking the convergence criterion.



7.7) Consider the 4-node element below:

a) Suggest a suitable approximation?

$$\phi = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$

b) Parasitic terms?

Yes,  $\alpha_4 xy$  is parasitic since the third row in pascal's triangle is not filled.

c) Use the C-matrix method to obtain the element shape functions.

$$\bar{N} = [1 \ x \ y \ xy], \quad C = \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \\ 1 & x_3 & y_3 & x_3 y_3 \\ 1 & x_4 & y_4 & x_4 y_4 \end{bmatrix}$$

d) What is the value of the element shape function  $N_2^e$  at the nodes 1, 2 & 3.

The point with element form functions is that they equal to zero

at all nodes except at a specific one. Thusly:

$$N_2^e(\text{node 1}) = 0, \quad N_2^e(\text{node 2}) = 1, \quad N_2^e(\text{node 3}) = 0$$

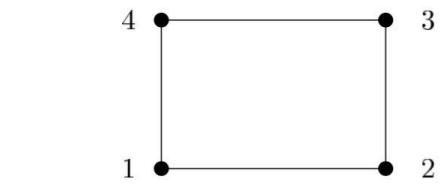
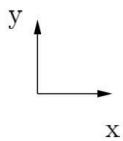


Figure 5: Four node element.