

6 - Strong and Weak Form of 2D/3D Heat Flow

6.1) An elliptic disc is defined by

$$g(x, y) = \left(\frac{2x-L}{2L}\right)^2 + \left(\frac{2y}{3L}\right)^2 \leq 1$$

The temperature field within the disc is given by

$$T = T_0 \left[\left(\frac{x}{3L}\right)^2 + \left(\frac{y}{L}\right)^2 \right]$$

The constitutive law is given by

$$\bar{q} = -\bar{D} \nabla T, \quad \bar{D} = k \bar{I}$$

a) Determine the heat flux vector at $(\frac{L}{2}, \frac{3L}{2})$.

$$\bar{\nabla} T = T_0 \left(\frac{2x}{3L^2}, \frac{2y}{L^2} \right)^T = \frac{2T_0}{L^2} \left(\frac{1}{3}x, y \right)^T$$

$$\bar{q} = -\bar{D} \bar{\nabla} T = -k \bar{I} \cdot \bar{\nabla} T = -\frac{2kT_0}{L^2} \left(\frac{1}{3}x, y \right)^T$$

$$\bar{q} \left(\frac{L}{2}, \frac{3L}{2} \right) = -\frac{kT_0}{L} \left(\frac{1}{3}, 3 \right)^T$$

b) Determine the normal vector at $(\frac{L}{2}, \frac{3L}{2})$

$$\bar{n} = \frac{\nabla g}{\|\nabla g\|} = \frac{\left(2\left(\frac{2x-L}{2L}\right)2, 8y/9L^2 \right)^T}{\left\| \left(2\left(\frac{2x-L}{2L}\right)2, 8y/9L^2 \right) \right\|}$$

$$\bar{n} \left(\frac{L}{2}, \frac{3L}{2} \right) = \frac{\left(0, \frac{8}{9L^2} \cdot \frac{3L}{2} \right)^T}{\left\| \left(0, \frac{8}{9L^2} \cdot \frac{3L}{2} \right) \right\|} = \frac{\left(0, \frac{4}{3L} \right)^T}{\left\| \left(0, \frac{4}{3L} \right) \right\|} = \frac{\frac{4}{3L} (0, 1)^T}{\frac{4}{3L} \|(0, 1)\|} = (0, 1)^T$$

c) Determine the heat flux.

$$q_n = q^T \bar{n} = -\frac{kT_0}{L} \left(\frac{1}{3}, 3 \right) \cdot (0, 1)^T = -\frac{3kT_0}{L}$$

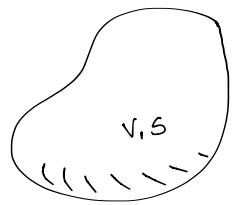
6.2) Q: supplied heat, q_n : heat flux leaving the body

a) Derive the global heat balance for the stationary situation.

Read page 81. The amount of supplied heat per unit equals the amount of heat leaving the body per unit time.

The supplied heat has to do with the volume while the leaving heat has to do with the surface area. Thus:

$$\int_S q_n dS = \int_V Q dV$$



b) Use Gauss divergence theorem to establish the strong form.

We can rewrite the left side of the eq. above.

$$\int_S q_n dS = \int_S \vec{q}^T \vec{n} dS \xrightarrow{\text{Gauss}} \int_V \operatorname{div}(\vec{q}) dV$$

$$\Rightarrow \int_V \operatorname{div}(\vec{q}) dV - \int_V Q dV = 0 \Leftrightarrow \boxed{\operatorname{div}(\vec{q}) - Q = 0}$$

Strong form

c) Derive the weak form.

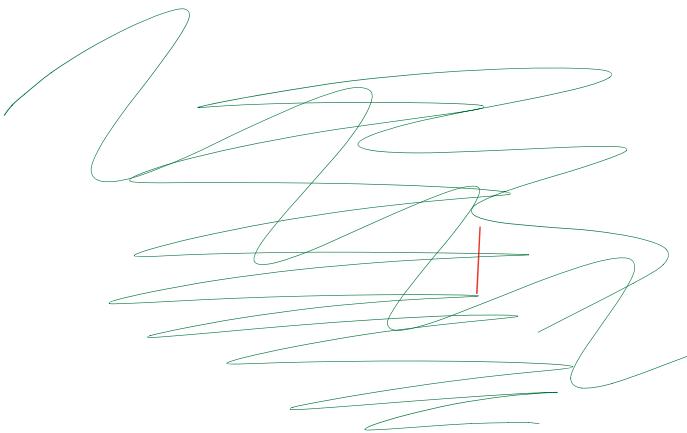
Multiply by a weight function v and integrate:

$$\int_V v \cdot \operatorname{div}(\vec{q}) dV - \int_V v Q dV = 0$$



Gauss and partial integration gives:

$$\int_V v \operatorname{div}(\vec{q}) dV = \int_S v \vec{q}^T \vec{n} dS$$



6.3) Isotropic materials: $\bar{q} = -\bar{D} \nabla T$,

$$\bar{D} = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix} = k I$$

Show that \bar{q} is parallel to ∇T

$$\bar{q} = -\bar{D} \cdot \nabla T = -k I \nabla T = -k \nabla T \quad \square$$

6.4) Fourier's law: $\bar{q}^T \cdot \nabla T \leq 0 \quad \forall \nabla T \neq 0$

Show that D^T exists.

D^T exists $\Leftrightarrow \det(D) \neq 0$

$$q^T \cdot \nabla T \leq 0$$

$$\Leftrightarrow (-\bar{D} \nabla T)^T \cdot \nabla T \leq 0$$

$$\Leftrightarrow -(\nabla T)^T \cdot \bar{D}^T \cdot (\nabla T) \leq 0$$

$$\Leftrightarrow (\nabla T)^T \cdot D^T \cdot (\nabla T) \geq 0 \quad \forall \nabla T$$

This means that D^T is positive definite $\Rightarrow D$ is positive definite $\Rightarrow \det(D) \neq 0$

\Rightarrow D^T exists

6.5) Newton convection is given by: $q_n = \alpha(T - T_\infty)$.

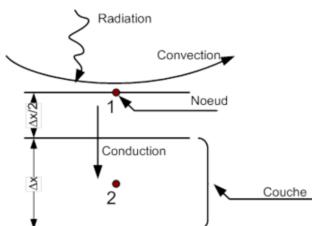
What is the mechanical analogy to this boundary condition?

A spring, $F = k(u - u_0)$

Convection thermique

■ Loi de Newton:

$$Q_{conv} = hA(T_\infty - T_1)$$



- h = constante de convection $W/(m^2 \cdot K)$;
- A = Section (m^2).

Radiation thermique

