

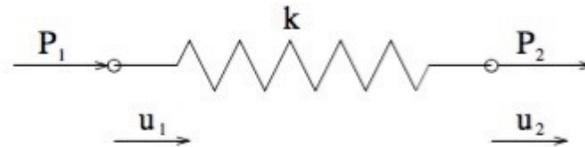
3-Direct method

3.1) a) Derive the stiffness matrix for a spring with stiffness K .

Spring eq: $F = k\delta$

$$\begin{cases} P_1 = -F = -K(u_2 - u_1) \\ P_2 = F = K(u_2 - u_1) \end{cases}$$

Matrix form: $\underbrace{\begin{bmatrix} K & -K \\ -K & K \end{bmatrix}}_{\text{stiffness matrix!}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$

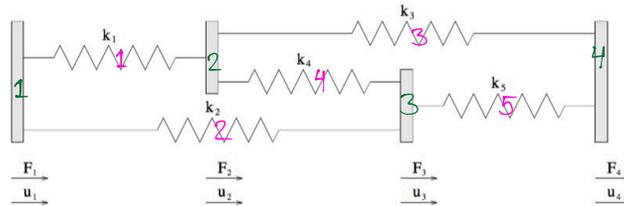


b) Derive the global stiffness matrix K for the following system:

We already have the element stiffness matrix (for one spring)

$$K_i^e = \begin{bmatrix} k_i & -k_i \\ -k_i & k_i \end{bmatrix}$$

To get K , we need to describe how the springs attach to each other by using Edof.



Edof = $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 4 \\ 4 & 2 & 3 \\ 5 & 3 & 4 \end{bmatrix}$, we insert five K_i^e according to Edof.

$$\Rightarrow K = \begin{bmatrix} K_1 + K_2 & -K_1 & -K_2 & 0 \\ -K_1 & K_1 + K_3 + K_4 & -K_4 & -K_3 \\ -K_2 & -K_4 & K_2 + K_4 + K_5 & -K_5 \\ 0 & -K_3 & -K_5 & K_3 + K_5 \end{bmatrix}$$

c) Give the physical interpretation of $\det(K) = 0$

$\det(K) = 0 \Leftrightarrow K$ is not invertible \Leftrightarrow There is no unique solutions to $KX = Y$.

d) Calculate u_2, u_3, F_1 and F_4 .

(Key: Apparently this means that "rigid body motion" is not prevented, all extensions are relative to each other, no boundary conditions)

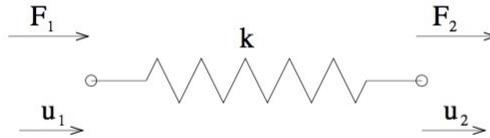
$u_1 = 1, u_4 = 0, F_2 = 0, F_3 = 20, k_1 = k_2 = k_3 = k_4 = k_5 = 8$

$$\Rightarrow 8 \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} F_1 \\ 0 \\ 20 \\ F_4 \end{bmatrix} \Leftrightarrow \begin{cases} 2 - u_2 - u_3 = F_1 \\ -1 + 3u_2 - u_3 = 0 \\ -1 - u_2 + 3u_3 = 20 \\ -u_2 - u_3 = F_4 \end{cases} \Rightarrow \begin{bmatrix} u_2 = 0,81 \text{ mm} \\ u_3 = 1,44 \text{ mm} \\ F_1 = -2 \text{ N} \\ F_4 = -18 \text{ N} \end{bmatrix}$$

3.2) a) Derive the element stiffness matrix.

Same method as in 3.1 a)

$$K_i^e = \begin{bmatrix} k_i & -k_i \\ -k_i & k_i \end{bmatrix}, K_i^e \alpha = f$$

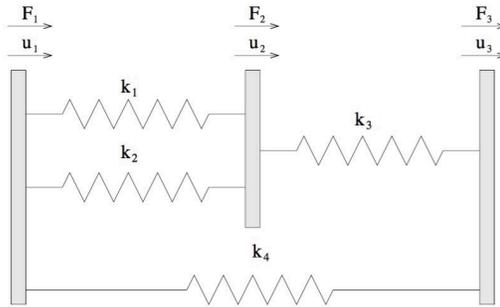


b) Derive the global load-displacement for the assembly below:

$$Edof = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}^T$$

$$\Rightarrow K = \begin{bmatrix} k_1+k_2+k_4 & -k_1-k_2 & -k_4 \\ -k_1-k_2 & k_1+k_2+k_3 & -k_3 \\ -k_4 & -k_3 & k_3+k_4 \end{bmatrix}$$

$$\alpha = [u_1 \ u_2 \ u_3]^T, f = [F_1 \ F_2 \ F_3]^T$$



$$K\alpha = f$$

c) Derive blabla for $F_2=0$... etc.

No.

d) Find an $\alpha \neq 0$ such that $\alpha^T K \alpha = 0$

$$\begin{cases} (k_1+k_2+k_4)a_1 - (k_1+k_2)a_2 - k_4a_3 = 0 \\ -(k_1+k_2)a_1 + (k_1+k_2+k_3)a_2 - k_3a_3 = 0 \\ -k_4a_1 - k_3a_2 + (k_3+k_4)a_3 = 0 \end{cases}, \text{ obvious solution: } \alpha = c \cdot [1 \ 1 \ 1]^T, c \text{ is a constant.}$$

3.3) Assembly of electrical resistances.

Ohms Law:

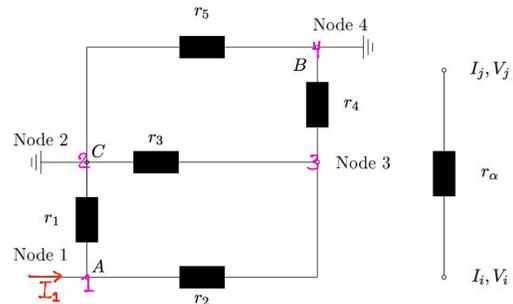
$$\begin{bmatrix} I_1^e \\ I_2^e \\ I_3^e \end{bmatrix} = \frac{1}{R} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} V_1^e \\ V_2^e \end{bmatrix} \quad I = \frac{U}{R}$$

Establish an equation system $K\alpha = f$.

$$Edof = \begin{bmatrix} 2 & 3 & 3 & 4 & 4 \\ 1 & 1 & 2 & 3 & 2 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}^T$$

$r_1=r_2=r_3=r_4=r_5=r^*$ gives the following matrix:

$$\Rightarrow K = \frac{1}{r^*} \begin{bmatrix} 1+1 & -1 & -1 & 0 \\ -1 & 1+1 & -1 & -1 \\ -1 & -1 & 1+1 & -1 \\ 0 & -1 & -1 & 1+1 \end{bmatrix} = \frac{1}{r^*} \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$



Solve the equation system.

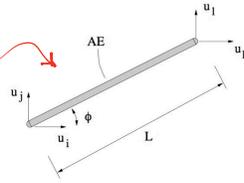
Remember linear algebra?

Good, here is the answer:

$$V_1 = 0,6r^*, \quad V_3 = 0,2r^*$$

3.4) Stiffness matrix for a bar:

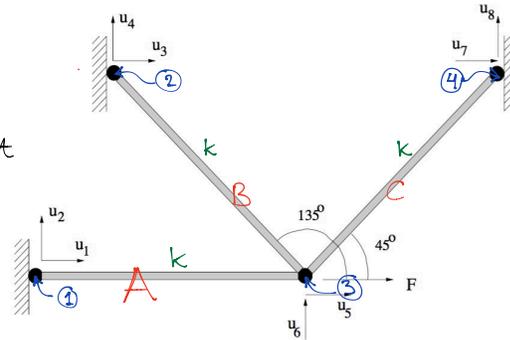
$$K^e = \frac{AE}{L} \begin{bmatrix} \cos^2 \phi & \sin \phi \cos \phi & -\cos^2 \phi & -\sin \phi \cos \phi \\ \sin \phi \cos \phi & \sin^2 \phi & -\sin \phi \cos \phi & -\sin^2 \phi \\ -\cos^2 \phi & -\sin \phi \cos \phi & \cos^2 \phi & \sin \phi \cos \phi \\ -\sin \phi \cos \phi & -\sin^2 \phi & \sin \phi \cos \phi & \sin^2 \phi \end{bmatrix}$$



Use this result to establish the global stiffness matrix for the system below. Assume $k = \frac{AE}{L}$ is the same for all three elements.

Let us begin by determining the element stiffness matrices for A, B and C.

Bar A ($\phi=0 \Rightarrow \cos \phi=1, \sin \phi=0$)



$$K^A = \frac{AE}{L} \begin{bmatrix} \cos^2 0 & \sin 0 \cos 0 & -\cos^2 0 & -\sin 0 \cos 0 \\ \sin 0 \cos 0 & \sin^2 0 & -\sin 0 \cos 0 & -\sin^2 0 \\ -\cos^2 0 & -\sin 0 \cos 0 & \cos^2 0 & \sin 0 \cos 0 \\ -\sin 0 \cos 0 & -\sin^2 0 & \sin 0 \cos 0 & \sin^2 0 \end{bmatrix} = k \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We do the same thing to get K^B and K^C .

$$K^A = k \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad K^B = \frac{k}{2} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \quad K^C = \frac{k}{2} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

Now we want to combine these three matrices, we do this by dividing them into 2×2 -sized submatrices (as indicated above). We can then construct a Edof matrix which tells us how to arrange our submatrices. We call the nodes 1, 2, 3 and 4.

$$Edof = \begin{bmatrix} A & 1 & 3 \\ B & 2 & 3 \\ C & 3 & 4 \end{bmatrix}$$

↖ which bar
↖ from to

$$\Rightarrow K = \begin{bmatrix} K_{11}^A & 0 & -K_{12}^A & 0 \\ 0 & K_{11}^B & -K_{12}^B & 0 \\ -K_{21}^A & -K_{21}^B & K_{22}^A + K_{22}^B + K_{11}^C & -K_{12}^C \\ 0 & 0 & -K_{21}^C & K_{22}^C \end{bmatrix} = \frac{k}{2} \begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ -2 & 0 & -1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 \end{bmatrix}$$