2 - Basic linear algebra

VT2016

2.1) The matrix K is defined by

$$K = \alpha B^T D B$$

where α is a scalar and B is 3×6.

a) Dimension of K?

$\dim(B) = 3 \times 6 \Leftrightarrow \dim(B^T) = 6 \times 3$

What is $\dim(D) = a \times b$?

Generally, if we want to do a matrix multiplication: $\underbrace{(x_1 \times y_1)}_{\dim(M_1)} \cdot \underbrace{(x_2 \times y_2)}_{\dim(M_2)}$,

then $y_1 = x_2$ and $\dim(M_1 \cdot M_2) = x_1 \times y_2$.

$\underbrace{(6\times 3)}_{\dim(B^T)} \cdot \underbrace{(a \times b)}_{\dim(D)} \cdot \underbrace{(3 \times 6)}_{\dim(B)} \Longrightarrow a = 3,\ b = 3$ and $\dim(B^T D B) = 6 \times 6$

Answer

Thus: $\boxed{\dim(K) = 6 \times 6}$

b) Determine dim(D)?

See above

$\boxed{\det(D) = a \times b = 3 \times 3}$

c) If $D = D^T$, show that K is symmetric.

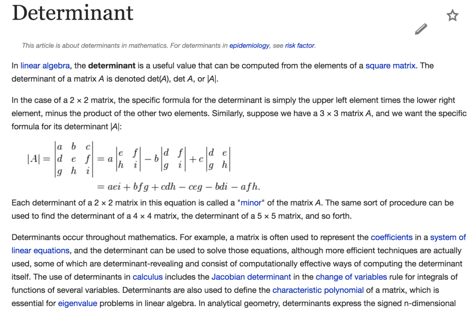
K is symmetric $\Leftrightarrow K = K^T$

$\boxed{K^T = (\alpha B^T D B)^T = \alpha B^T D^T (B^T)^T = \alpha B^T D\ B = K.}$ $\square$

2.2) Calculate $\det(K)$.

$$\det(K) = \begin{vmatrix} 1 & 6 & 2 & 3 \\ 0 & 2 & 0 & 0 \\ 1 & 6 & -2 & 1 \\ 0 & 3 & 1 & 2 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 1 & -2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 2(-4+0+3-0-1-4) = \boxed{-12}$$

Answer



Determinant

This article is about determinants in mathematics. For determinants in epidemiology, see risk factor.

In linear algebra, the **determinant** is a useful value that can be computed from the elements of a square matrix. The determinant of a matrix *A* is denoted det(*A*), det *A*, or |*A*|.

In the case of a 2 × 2 matrix, the specific formula for the determinant is simply the upper left element times the lower right element, minus the product of the other two elements. Similarly, suppose we have a 3 × 3 matrix *A*, and we want the specific formula for its determinant |*A*|:

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$
$$= aei + bfg + cdh - ceg - bdi - afh.$$

Each determinant of a 2 × 2 matrix in this equation is called a "minor" of the matrix *A*. The same sort of procedure can be used to find the determinant of a 4 × 4 matrix, the determinant of a 5 × 5 matrix, and so forth.

Determinants occur throughout mathematics. For example, a matrix is often used to represent the coefficients in a system of linear equations, and the determinant can be used to solve those equations, although more efficient techniques are actually used, some of which are determinant-revealing and consist of computationally effective ways of computing the determinant itself. The use of determinants in calculus includes the Jacobian determinant in the change of variables rule for integrals of functions of several variables. Determinants are also used to define the characteristic polynomial of a matrix, which is essential for eigenvalue problems in linear algebra. In analytical geometry, determinants express the signed n-dimensional

Not trivial

What is a determinant and why can we "expand along 2nd row"?

2.3) $M = a^T K a$, $K = K^T$, $\dim(K) = n \times n$, $\dim(a) = n \times 1$, $a^T K a \geqslant 0$, equality holds for some $a \neq 0$

a) Determine $\det(K)$

We know that K is positive semi-definite.

Algebra > Linear Algebra > Matrices > Matrix Types >

Positive Semidefinite Matrix

A positive semidefinite matrix is a Hermitian matrix all of whose eigenvalues are nonnegative.

Positive-definite matrix

Not to be confused with Positive matrix and Totally positive matrix.

In linear algebra, a symmetric $n \times n$ real matrix M is said to be **positive definite** if the scalar $z^T M z$ is positive for every non-zero column vector z of n real numbers. Here z^T denotes the transpose of z.[1]

More generally, an $n \times n$ Hermitian matrix M is said to be **positive definite** if the scalar $z^* M z$ is real and positive for all non-zero column vectors z of n complex numbers. Here z^* denotes the conjugate transpose of z.

The **negative definite**, **positive semi-definite**, and **negative semi-definite** matrices are defined in the same way, except that the expression $z^T M z$ or $z^* M z$ is required to be always negative, non-negative, and non-positive, respectively.

Positive definite matrices are closely related to positive-definite symmetric bilinear forms (or sesquilinear forms in the complex case), and to inner products of vector spaces.[2]

Some authors use more general definitions of "positive definite" that include some non-symmetric real matrices, or non-Hermitian complex ones.

We know that $\det(K) = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n$

where λ_k, $k = 1, \ldots, n$ are eigenvalues to K.

So how can we determine the eigenvalues $\{\lambda_i\}$?

$Kv = \lambda v \Rightarrow v^T K v = v^T v \lambda$

We know that $v^T K v \geqslant 0$

$\Rightarrow v^T v \lambda \geqslant 0$

It is also known that $v^T v = [v_1^* \; v_2^* \ldots] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \end{bmatrix} = |v_1|^2 + |v_2|^2 + \ldots > 0 \Rightarrow \boxed{\lambda \geqslant 0}$

Given: $x^T K x = 0$ for some $x \neq 0 \Rightarrow$ At least one eigenvalue is equal to zero.

Answer

Thus, $\boxed{\det(K) = 0}$

b) Does $Kx = 0$ have non-trivial solutions?

$\boxed{\text{Yes}}$, example: $K = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Kernel (linear algebra)

Page issues

In mathematics, and more specifically in linear algebra and functional analysis, the **kernel** (also known as **null space** or **nullspace**) of a linear map $L : V \to W$ between two vector spaces V and W, is the set of all elements $\mathbf{v}$ of V for which $L(\mathbf{v}) = \mathbf{0}$, where $\mathbf{0}$ denotes the zero vector in W. That is, in set-builder notation,

$$\ker(L) = \{\mathbf{v} \in V | L(\mathbf{v}) = \mathbf{0}\}.$$

c) $b \neq 0$, How many solutions to $Kx = b$ does exist?

example: $K = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

We see here that no solutions exist.

example: $K = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

We see here that ∞ solutions exist, $x = \begin{bmatrix} 1 \\ c \end{bmatrix}$ where c can be chosen arbitrarily.

Can there exist a finite number of solutions?

No, think about why. $(\det(A) = 0)$

$\boxed{\text{Answer: No solutions or } \infty \text{ solutions.}}$

KERNEL AND IMAGE OF A MATRIX

Take an $n \times m$ matrix

or $Mx = 0$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

and think of it as a function

$$A : \mathbb{R}^m \longrightarrow \mathbb{R}^n.$$

The **kernel** of A is defined as

$$\ker A = \text{set of all } \mathbf{x} \text{ in } \mathbb{R}^m \text{ such that } A\mathbf{x} = \mathbf{0}.$$

Note that $\ker A$ lives in $\mathbb{R}^m$.

The **image** of A is

$$\text{im } A = \text{set of all vectors in } \mathbb{R}^n \text{ which are } A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^m.$$

2.4) $T = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3$.

Fit the parameters α_i to the measured data.

i	T_i	x_i
1	0.31	0.12
2	0.32	0.15
3	0.34	0.16
4	0.36	0.19

$T_i = \alpha_1 + \alpha_2 x_i + \alpha_3 x_i^2 + \alpha_4 x_i^3$

Matrix form:

$$\underbrace{\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix}}_{T} = \underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_1 & x_1^2 & x_1^3 \end{bmatrix}}_{X} \cdot \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}}_{\bar{\alpha}}$$

$$\bar{\alpha} = X^{-1} T = \boxed{[0{,}0041 \quad -0{,}0765 \quad 0{,}5024 \quad -1{,}0714]^T \cdot 10^3}$$

2.5) $\bar{v} = [1 \ 3 \ 2]^T$ m/s, $A = 0{,}2$ m², $\bar{n} = \frac{1}{2}[\sqrt{3} \ 1 \ 0]^T$

Calculate the amount of water passing the surface per second.

We want to multiply each component in $\bar{v}$ with the corresponding one in $\bar{n}$ and also with the area A.

$$\Rightarrow \text{Answer} = A \cdot \bar{v}^T \bar{n} = 0.2 \cdot [1 \ 3 \ 2] \begin{bmatrix} \sqrt{3} \\ 1 \\ 0 \end{bmatrix} \cdot \frac{1}{2} = 0{,}2\,(\sqrt{3} + 3 + 0) \cdot \frac{1}{2} = \boxed{0{,}4732 \ m^3/s}$$

2.6) Calculate u_3, u_4, f_1 and f_2.