10- FE formulation of 3D heat flow

10.1) A chimney with three channels.

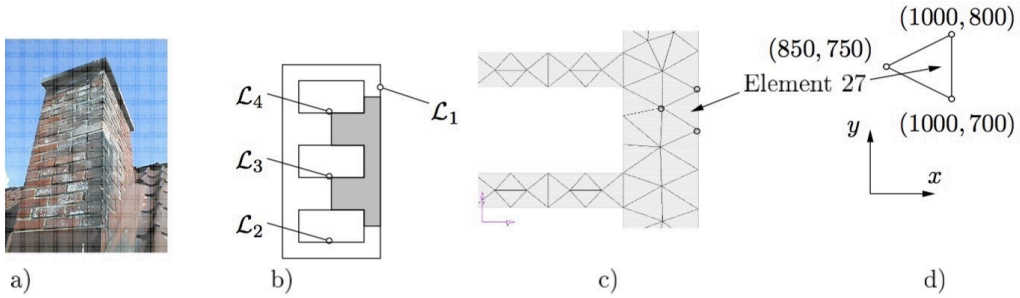


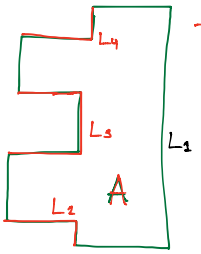
Figure 7: a) Chimney. b) Cross-section of chimney. c) FE-mesh of element 27 and its surrounding elements. d) Location of element 27

$div(\underline{q}) = 0$, $\underline{q} = -k \nabla T$

$L_1: q_n = \alpha(T-22)$, $L_2: T = 300$, $L_3: q_n = \alpha(T-22)$, $L_4: q_n = \alpha(T-22)$

a) Derive the FE formulation

Mult. by v and integrate.



The interesting piece.

$$\int_A v\, t \nabla(\underline{q})\, dA = 0$$

$$\Leftrightarrow \oint_L v\, t\, \underbrace{\underline{q}^T \underline{n}}_{q_n}\, dL - \int_A (\nabla v)^T \underline{q}\, t\, dA = 0$$

$$\Leftrightarrow \int_{L_1} + \int_{L_2} + \int_{L_3} + \int_{L_4} - \int_A$$

Use $v = \mathbb{C}^T \mathbb{N}^T$, $\mathbb{B} = \nabla \mathbb{N}$, $T = \mathbb{N}a \Rightarrow \nabla T = \nabla \mathbb{N} a = \mathbb{B} a$

$$\oint_L \mathbb{C}^T \mathbb{N}^T t\, q_n\, dL - \int_A \underbrace{\mathbb{C}^T \mathbb{B}^T}_{(\nabla v)^T} \underbrace{(-k \mathbb{B} a)}_{\underline{q}} dA = 0$$

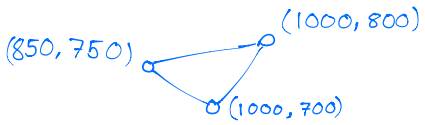
$$\Leftrightarrow \mathbb{C}^T \left(\oint_L \mathbb{N}^T t\, q_n\, dL + \int_A \mathbb{B}^T \mathbb{B}\, k\, a\, dA \right) = 0$$

FE formulation.

$$\underbrace{\int_A \mathbb{B}^T \mathbb{B} k\, dA}_{\mathbb{K}}\, a = \underbrace{-\oint_L \mathbb{N}^T t\, q_n\, dL}_{f} \quad \Leftrightarrow \quad \boxed{\mathbb{K} a = f}$$

b) Calculate the element matrices for elemen 27.

C-matrix method



$$\boxed{\mathbf{a} = \mathbf{C}\boldsymbol{\alpha}} \iff \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 1 & 850 & 750 \\ 1 & 1000 & 700 \\ 1 & 1000 & 800 \end{bmatrix}, \quad \mathbf{C}^{-1} = \begin{bmatrix} \frac{20}{3} & \frac{14}{3} & -\frac{31}{3} \\ -\frac{1}{150} & \frac{1}{300} & \frac{1}{300} \\ 0 & -\frac{1}{100} & \frac{1}{100} \end{bmatrix}$$

We now want to determine $\mathbb{N}^e$.

$$\begin{cases} T = \mathbb{N}\alpha \\ \mathbf{a} = \mathbf{C}\boldsymbol{\alpha} \end{cases} \Rightarrow T = \underbrace{\mathbb{N}\mathbf{C}^{-1}}_{\mathbb{N}^e}\mathbf{a} = \mathbb{N}^e\mathbf{a} \qquad \text{where } \mathbb{N} = [1 \;\; x \;\; y]$$

$$\mathbb{N}^e = [\mathbb{N}_1^e \;\; \mathbb{N}_2^e \;\; \mathbb{N}_3^e]^T = [1 \;\; x \;\; y] \begin{bmatrix} \frac{20}{3} & \frac{14}{3} & -\frac{31}{3} \\ -\frac{1}{150} & \frac{1}{300} & \frac{1}{300} \\ 0 & -\frac{1}{100} & \frac{1}{100} \end{bmatrix} = \boxed{\begin{bmatrix} \frac{20}{3} - \frac{1}{150}x \\ \frac{14}{3} + \frac{1}{300}x - \frac{1}{100}y \\ \frac{31}{3} + \frac{1}{300}x + \frac{1}{100}y \end{bmatrix}^T} \; \mathbb{N}^e$$

Hmm, what is $\mathbb{B}^e$?

$$\mathbb{B}^e = \nabla^T\mathbb{N}^e = \begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix} \cdot \begin{bmatrix} \frac{20}{3} - \frac{1}{150}x \\ \frac{14}{3} + \frac{1}{300}x - \frac{1}{100}y \\ \frac{31}{3} + \frac{1}{300}x + \frac{1}{100}y \end{bmatrix}^T = \begin{bmatrix} \frac{1}{150} & \frac{1}{300} & \frac{1}{300} \\ 0 & -\frac{1}{100} & \frac{1}{100} \end{bmatrix} = \boxed{\frac{1}{300}\begin{bmatrix} 2 & 1 & 1 \\ 0 & -3 & 3 \end{bmatrix}}$$

Element stiffness matrix.

$$\mathbb{K}^e = \int_{A^e} \mathbb{B}^{eT}\mathbb{B}^e k \, dA = \mathbb{B}^{eT}\mathbb{B}^e k \int_{A^e} dA = \mathbb{B}^{eT}\mathbb{B}^e k A^e = \left(\frac{1}{300}\right)^2 \begin{bmatrix} -2 & 0 \\ 1 & -3 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \end{bmatrix} k A^e = \boxed{\begin{bmatrix} 4 & -2 & -2 \\ -2 & 10 & -8 \\ -2 & -8 & 10 \end{bmatrix} \frac{kA^e}{300^2}}$$

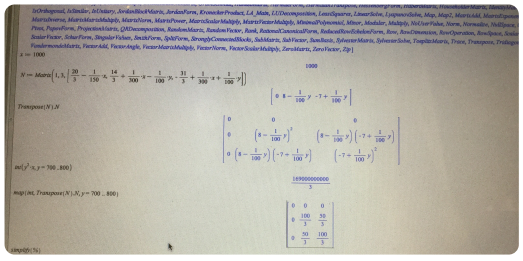
Use the triangle area theorem to determine A^e.

$$A^e = \frac{100 \cdot 150}{2} \Rightarrow \frac{A^e}{300^2} = \frac{100 \cdot 150}{300 \cdot 300 \cdot 2} = \frac{1}{3 \cdot 2 \cdot 2} = \frac{1}{12} \Rightarrow \boxed{\mathbb{K}^e = \frac{k}{12}\begin{bmatrix} 4 & -2 & -2 \\ -2 & 10 & -8 \\ -2 & -8 & 10 \end{bmatrix}}$$

Great! Well done!

$$\bar{f} = \int_{L_{23}} \mathbb{N}^{eT} q_n \, dL = \int_{L_{23}} \mathbb{N}^{eT}\alpha (T - T_\infty) dL = \int_{L_{23}} \mathbb{N}^{eT}\alpha \left(\mathbb{N}^e\alpha^e - T_\infty\right) dL =$$

$$= \alpha\left(\int_{L_{23}} \ldots \text{ FUCK THIS} \ldots\right) = \text{Answer} \; \boxed{\frac{\alpha}{6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\alpha^e - \frac{\alpha T_\infty}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}$$



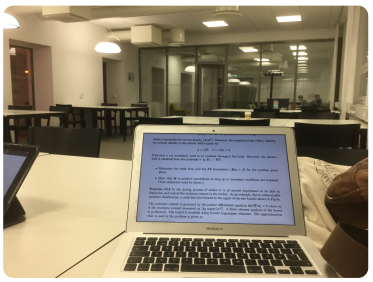
10.2) Blabla continous current.

$$\operatorname{div}(\bar{j}) = 0 \Leftrightarrow \nabla \bar{j} = 0$$

Ohm's law: $\bar{j} = \sigma \bar{E}$, $\sigma = \sigma(x) > 0$

$$E = -\nabla V$$

a) Determine the weak form and the FE-formulation.



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I am getting tired of this...

$$\int_V v \nabla \bar{j}^e dV = 0$$

$$\Leftrightarrow \oint_S v \underbrace{\bar{j}^T \bar{n}}_{j_n} ds - \int_V (\nabla v) \bar{j}^e dV = 0$$

Use $\bar{j} = \sigma E = \sigma \nabla V$

Weak form

$$\boxed{-\oint_S v j_n ds + \int_V (\nabla v) \sigma \nabla V dV}$$

⚠ Remark: $v \neq V \neq V$ (These V:s might be symbolically same, but they do not represent the same quantity.)

Galerkin: $V = C^T N^T$, $\nabla V = C^T B^T$ (NOT THE SAME!)

$v = Na$, $\nabla V = Ba$

$$\Rightarrow C^T \left(\int_V B^T \sigma B dV a + \oint_S N^T j_n ds \right) = 0$$

$$\Rightarrow \boxed{\int_V B^T \sigma B dV a = -\oint_S N^T j_n ds} \Leftrightarrow \boxed{Ka = f}$$ FE formulation

b) Show that K is positive semidefinite.

$$K = \int_V B^T B \sigma dV \quad \text{, show that } x^T K x \geq 0$$

Multiply by a^T from left and a from right

$$a^T K a = \int_V \underbrace{a^T B^T}_{(Ba)^T} B a \sigma dV = \int (Ba)^T (Ba) \sigma dV = \int \|Ba\|^2 \sigma dV = \int \|\nabla v\|^2 \sigma dV \geq 0 \quad \square$$

When does equality hold?

$$B = \nabla N = \nabla [N_1 \dots N_n]$$

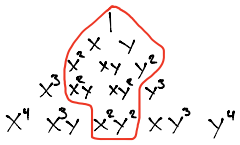
$$\Rightarrow Ba = \nabla [N_1 \dots N_n] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$a = [1 \dots 1]^T \Rightarrow Ba = \nabla \underbrace{(N_1 + \dots + N_n)}_{\text{The sum of all shape functions is equal to one.}} = \nabla(1) = 0 \quad \square$$

10.3) Moisture

$$\operatorname{div}(\nabla m) = 0$$

$$m = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^2 y + \alpha_8 x y^2 + \alpha_9 x^2 y^2$$



a) Parasitic terms?

Yes, $\alpha_7 x^2 y$, $\alpha_8 x y^2$ & $\alpha_9 x^2 y^2$. This means that the convergence order is not better than for a second degree polynomial (6-node element).

b) Does it converge?

Convergence = Completeness + compatibility

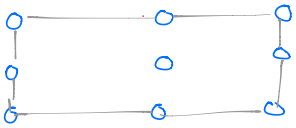
Completeness

Yes, since we have both x- and y-terms.

Compatibility

$$\left.\begin{aligned} x = k &\Rightarrow m = \beta_1 + \beta_2 y + \beta_3 y^2 \\ y = k &\Rightarrow m = \gamma_1 + \gamma_2 x + \gamma_3 x^2 \end{aligned}\right\}$$

Since we are looking at the boundaries, we can set x (or y) to a constant.



We then have three points which must be satisfied, if we want to have unique solution, then m must be a second degree polynomial!

The solution converges!

c) Calculate the contribution from the boundary term at nodes 1, 2 & 3.

$$\int_{L_{123}} \mathbf{N}^T q_n \, dL$$

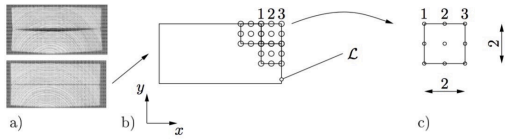
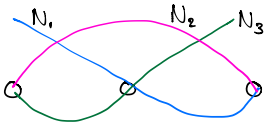


Figure 8: a) Two boards that have been dried under different conditions. b) The boundary and three elements that are used in the finite element analysis of the moisture problem. c) One 9-node Lagrangian element.

We need to derive $\mathbf{N}$ when $y = 2$.

$$\mathbf{N} = [N_1 \; N_2 \; N_3 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0]$$



$$\begin{cases} N_1 = (x-1)(x-2)\cdot\frac{1}{2} \\ N_2 = -x(x-2) \\ N_3 = x(x-1)\cdot\frac{1}{2} \end{cases}$$

$$\Rightarrow \int_0^2 [N_1 \; N_2 \; N_3 \; 0 \; \dots] q_0 \, dx = \left[q_0 \begin{bmatrix} \frac{1}{6}x^3 - \frac{3}{4}x^2 + x \\ -\frac{1}{3}x^3 + x^2 \\ \frac{1}{6}x^3 - \frac{1}{4}x^2 \end{bmatrix} \right]_0^2 = q_0 \begin{bmatrix} 1/3 \\ 4/3 \\ 1/3 \end{bmatrix}$$

VF SKA DET BLI GGR 2 FÖR SISTA?

10.4) Diffusion problem.

$$\operatorname{div}(\nabla c) = 0$$

$$\frac{\partial c}{\partial x} = 0 \quad \text{along} \quad x = 0; \qquad \frac{\partial c}{\partial x} = 1 \quad \text{along} \quad x = 1$$

$$c = 1 \quad \text{along} \quad y = 0; \qquad \frac{\partial c}{\partial y} + c = 2 \quad \text{along} \quad y = 1$$

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a) Derive the FE formulation.

$$\int_A v \operatorname{div}(\nabla c)\, dS = 0$$

$$\Leftrightarrow \oint_L v (\nabla c)^T \bar{n}\, dL - \int_A (\nabla v)^T \cdot (\nabla c)\, dS = 0$$

Gelarkin: $V = C^T N^T$, $[\nabla v]^T = C^T B^T$

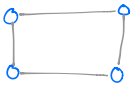
$c = Na$, $\nabla c = Ba$

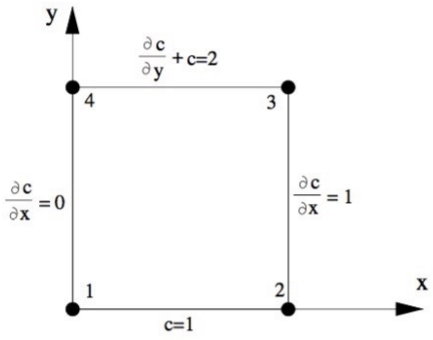
$$\Rightarrow C^T \left(\oint_L N^T (\nabla c)^T \bar{n}\, dL - \int_A B^T B\, dS\, a \right) = 0$$

$$\Rightarrow \oint_L N^T (\nabla c)^T \bar{n}\, dL - \int_A B^T B\, dS\, a = 0 \Leftrightarrow \boxed{Ka = f}$$

b) Determine the concentration along $y = 1$.

Four node element:





We would really like to find the element form functions.

Lagrange

$$c = \underbrace{\frac{(x-1)(y-1)}{(0-1)(0-1)}}_{N_1^e} + \underbrace{\frac{x(y-1)}{(0-1)}}_{N_2^e} + \underbrace{x \cdot y}_{N_3^e} + \underbrace{\frac{(x-1) \cdot y}{(0-1)}}_{N_4^e} =$$

$$= (x-1)(y-1) - x(y-1) + xy - (x-1) \cdot y$$

$$\Rightarrow N^e = \begin{bmatrix} (x-1)(y-1) & -x(y-1) & xy & -(x-1) \cdot y \end{bmatrix}$$

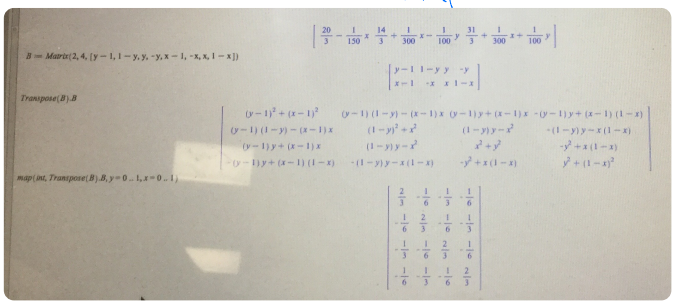
$$B^e = \nabla N^e = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} (x-1)(y-1) & -x(y-1) & xy & -(x-1) \cdot y \end{bmatrix} = \boxed{\begin{bmatrix} y-1 & 1-y & y & -y \\ x-1 & -x & x & 1-x \end{bmatrix}} \; B^e$$

Scroll to the next page to find K.

Finding $\mathbb{K}$.

$$\mathbb{K} = \int_A \mathbb{B}^T \mathbb{B}\, dS = \int_0^1\int_0^1 \mathbb{B}^T\mathbb{B}\, dx dy = \dots = \dots = \dots = \dots = \frac{1}{6}\begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{bmatrix}$$

Maple



Finding $\mathbb{f}$.

$$\mathbb{f} = \oint_L \mathbb{N}^T (\nabla c)^T \hat{n}\, dL = \int_{L_{12}} + \int_{L_{23}} + \int_{L_{34}} + \int_{L_{41}}$$

$$\mathbb{N}^T = \begin{bmatrix} (1-x)(1-y) & -x(y-1) & xy & -(x-1)\cdot y \end{bmatrix}^T$$

$$\int_{L_{12}} = \int_0^1 \begin{bmatrix} 1-x & x & 0 & 0 \end{bmatrix}^T \underbrace{(\nabla 1)}_{=0} (-\hat{y})\, dx = \boxed{0}\ L_{12}$$

$$\begin{bmatrix} \frac{\partial c}{\partial x} & \frac{\partial c}{\partial y} & \frac{\partial c}{\partial z} \end{bmatrix} = \begin{bmatrix}$$

$$\int_{L_{23}} = \int_0^1 \begin{bmatrix} 0 & 1-y & y & 0 \end{bmatrix}^T (\nabla c)\hat{x}\, dy = \int_0^1 \begin{bmatrix} 0 & 1-y & y & 0 \end{bmatrix}^T 1\, dy = \begin{bmatrix} 0 & 1-\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}^T = \boxed{\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}^T}\ L_{23}$$

$$\int_{L_{34}} = \int_0^1 \begin{bmatrix} 0 & 0 & x & 1-x \end{bmatrix} (\nabla c)^T \cdot \hat{y}\, dx$$

What about ∇c?

$$\frac{dc}{dy} + c = 2 \iff \frac{\partial c}{\partial y} = 2 - c = 2 - \mathbb{N}\mathbb{a}$$

$$\Rightarrow (\nabla c) = \left(\frac{\partial c}{\partial x}, 2 - \mathbb{N}\mathbb{a}\right)$$

$$\Rightarrow (\nabla c)^T \cdot \hat{y} = 2 - \mathbb{N}\mathbb{a}$$

$$\Rightarrow \int_{L_{34}} = \int_0^1 \mathbb{N}^T (2 - \mathbb{N}\mathbb{a})\, dx = \int_0^1 2\mathbb{N}^T - \mathbb{N}^T\mathbb{N}\mathbb{a}\, dx = \int_0^1 2\begin{bmatrix} 0 \\ 0 \\ x \\ 1-x \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & x^2 & x(1-x) \\ 0 & 0 & (1-x)x & (1-x)^2 \end{bmatrix} \mathbb{a}\, dx =$$

$$= \left[2\begin{bmatrix} 0 \\ 0 \\ \frac{x^2}{2} \\ x - \frac{x^2}{2} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{x^3}{3} & \frac{x^2}{2} - \frac{x^3}{3} \\ 0 & 0 & \frac{x^2}{2} - \frac{x^3}{3} & \frac{-(1-x)^3}{3} \end{bmatrix} \mathbb{a} \right]_0^1 = 2\cdot\begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 1/6 \\ 0 & 0 & 1/6 & 1/3 \end{bmatrix} \mathbb{a} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}}_{\text{Add this to the f-vector}} - \frac{1}{6}\overbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}}^{\mathbb{M}} \mathbb{a}$$

Add this to the $\mathbb{K}$ matrix.

$$\int_{L_{41}} = \int_1^0 \mathbb{N}^T (\nabla c)^T \hat{n}\, dx$$

$$\hat{n} = -\hat{x} \Rightarrow (\nabla c)^T \hat{n} = -\frac{\partial c}{\partial x}$$

$$\frac{\partial c}{\partial x} = 0 \Rightarrow (\nabla c)\cdot\hat{n} = 0 \Rightarrow \int_{L_{41}} = \boxed{0}\ L_{41}$$

Put everything together:

$$\oint = \int_{L_{12}} + \int_{L_{23}} + \int_{L_{34}} + \int_{L_{41}} = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ 1/2 \\ 3/2 \\ 1 \end{bmatrix}}$$

We had $\mathbb{K}\mathbb{a} = \mathbb{f}$ but since $\mathbb{f}$ seems to have a matrix part we now define:

$\boxed{\tilde{\mathbb{K}}\mathbb{a} = \tilde{\mathbb{f}}}$ where $\tilde{\mathbb{K}} = \mathbb{K} - \mathbb{M} = \frac{1}{6} \cdot \begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 6 & 0 \\ -1 & -2 & 0 & 6 \end{bmatrix}}$ $\mathbb{K}$

and $\mathbb{f} = \boxed{\begin{bmatrix} 0 \\ 1/2 \\ 3/2 \\ 1 \end{bmatrix}}$

$\tilde{\mathbb{K}}\mathbb{a} = \tilde{\mathbb{f}} \iff \mathbb{a} = \tilde{\mathbb{K}}^{-1}\tilde{\mathbb{f}} =$ /Maple/ =

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